New Results on Betting Strategies, Market Selection, and the Role of Luck

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Outline

Who's gonna win?

Figure: Cornelis de Vos, (1584–1651)

Agents repeatedly bet on uncertain outcome: the event $s_t \in \{0, 1\}$ is an independent Bernoulli trial with success probability π^*.

My house, my rules

Parimutuel system with no house take, p_t is the inverse odd ratio at round t: if $s_t = 1$, those who bet on the occureence receive $1/p_t$ times the amount bet. If $s_t = 0$ those who bet against the occurrence of the event receives $1/(1-p_t)$.

Bettors behavior

In each round, agent i has to choose the fraction of wealth to be wagered b_t^i and the side of the bet $\sigma_t^i \in \{0, 1\}$, where 1 means betting on the occurrence of the event while 0 betting against it. Decision based on prevailing odds.

For each agent i there exists a "fair" inverse odd $\bar{p}^i \in (0, 1)$ and a continuous function $b^i \in [0, 1)$ such that $\sigma_t^i = 1$ if $p_t < \bar{p}^i$, $\sigma_t^i = 0$ if $p_t > \bar{p}^i$, $b^i(\bar{p}^i) = 0$ and $b^i(p_t) > 0$ when $p_t \neq \bar{p}^i$ (Kets et al., 2014).

Two bettors

They take different side of the bet. Assume $\bar{p}^1 < \bar{p}^2$

$$\frac{w^1_{t-1} b^1(p_t)}{1-p_t} = \frac{w^2_{t-1} b^2(p_t)}{p_t} .$$

Wealth of agent i at time t evolve

$$w^i_t = \left(1 - b^i(p_t)\right) w^i_{t-1} + \delta_{s_t, i-1} w^i_{t-1} b^i(p_t) \left(\frac{\delta_{i,1}}{1-p_t} + \frac{\delta_{i,2}}{p_t} \right)$$

No house take: the aggregate wealth is constant and we set $w_t = w^1_t + w^2_t = 1$ such that $p_t \in [\bar{p}^1, \bar{p}^2]$ and $p_t = \bar{p}^i$ if and only if $w^i_t = 1$.

Previous results

If “odds” are exogenous, the strategy that guarantees optimal wealth growth is the Kelly rule (Kelly, 1956; Breiman, 1961).

In a population of Kelly bettors (endogenous odds), the one with the most accurate beliefs accrues all the wealth (Beygelzimer et al., 2012; Blume and Easley, 1992).

An agent adopting the Kelly strategy and having correct beliefs will accrue all the wealth irrespective of the strategies adopted by other agents (as far as they don’t depend on odds) (Evstigneev et al., 2002).

Kets et al. (2014) consider the case of two fractional Kelly and CRRA bettors, providing a few tentative results based on numerical simulations but nothing is proved analytically.

We provide analytical results about the two-bettor case with generic odd-dependent betting strategies.

Outline

Survival and Dominance

Agent i (asymptotically) *dominates* on σ if $\lim_{t\to\infty} w_t^i(\sigma) = 1$. Agent i (asymptotically) *survives* on σ if $\limsup_{t\to\infty} w_t^i(\sigma) > 0$.

Agent i (asymptotically) *dominates* if $\lim_{t\to\infty} w_t^i(\sigma) = 1$ for almost all σ. Agent i (asymptotically) *survives* if $\limsup_{t\to\infty} w_t^i > 0$ for almost all σ.

Main result

Consider

$$\mu^1 = \pi^* \log \frac{\bar{p}^1 + (1-\bar{p}^1) b^2(\bar{p}^1)}{\bar{p}^1} + (1-\pi^*) \log(1 - b^2(\bar{p}^1)) \quad (1)$$

and

$$\mu^2 = -\pi^* \log(1 - b^1(\bar{p}^2)) - (1-\pi^*) \log \frac{1 - \bar{p}^2 + \bar{p}^2 b^1(\bar{p}^2)}{1-\bar{p}^2} \,. \quad (2)$$

i) if $\mu^1 > 0$ and $\mu^2 > 0$ agent 2 dominates and $\lim_{t\to\infty} p_t = \bar{p}^2$ almost surely;

ii) if $\mu^1 < 0$ and $\mu^2 < 0$ agent 1 dominates and $\lim_{t\to\infty} p_t = \bar{p}^1$ almost surely;

iii) if $\mu^2 < 0$ and $\mu^1 > 0$ both agents survive;

iv) if $\mu^2 > 0$ and $\mu^1 < 0$ either agent 1 dominates or agent 2 dominates depending on the realization of the Bernoulli.

Outline

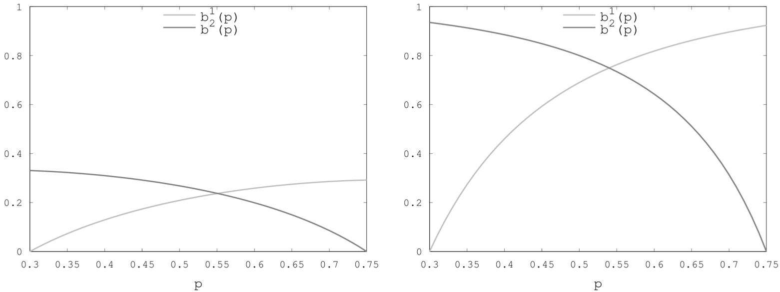
Expected utility maximizer

Maximize the expected utility of wealth using a power utility. Let $\gamma^i > 0$ be the relative risk aversion coefficient of agent i and π^i the subjective probability (belief) assigned to the realization of the event. Assuming $\pi^1 < \pi^2$, for $p_t \in [\pi^1, \pi^2]$ agent 1 bets against the occurrence of the event and agent 2 in favor.

$$b^1(p_t) = \frac{\left(p_t(1-\pi^1)\right)^{\frac{1}{\gamma^1}} - \left(\pi^1(1-p_t)\right)^{\frac{1}{\gamma^1}}}{\left(p_t(1-\pi^1)\right)^{\frac{1}{\gamma^1}} + p_t\left(\pi^1\right)^{\frac{1}{\gamma^1}}\left(1-p_t\right)^{\frac{1-\gamma^1}{\gamma^1}}}\,.$$

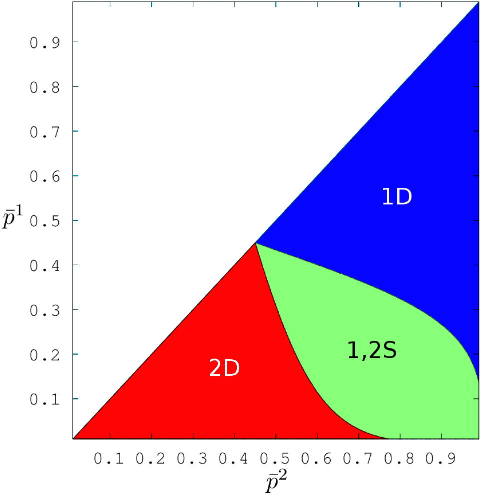
$$b^2(p_t) = \frac{\left(\pi^2(1-p_t)\right)^{\frac{1}{\gamma^2}} - \left(p_t(1-\pi^2)\right)^{\frac{1}{\gamma^2}}}{\left(\pi^2(1-p_t)\right)^{\frac{1}{\gamma^2}} + (1-p_t)\left(1-\pi^2\right)^{\frac{1}{\gamma^2}}\left(p_t\right)^{\frac{1-\gamma^2}{\gamma^2}}}\,.$$

Examples



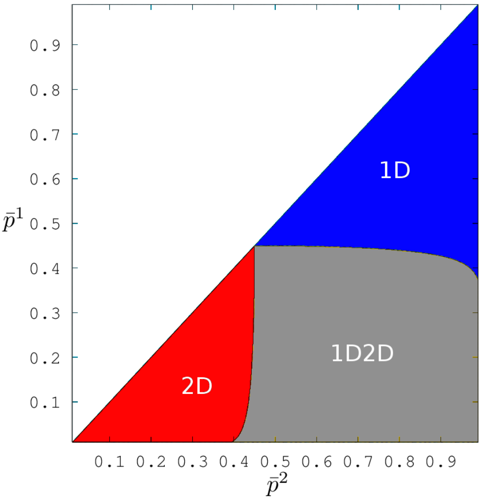
In both panels we set $\bar{p}^1 = 0.3$ and $\bar{p}^2 = 0.75$. Left: $\gamma^1 = \gamma^2 = 2$. Right: $\gamma^1 = \gamma^2 = 0.5$.

$\pi^* = 0.45, \gamma^1 = \gamma^2 = 2$



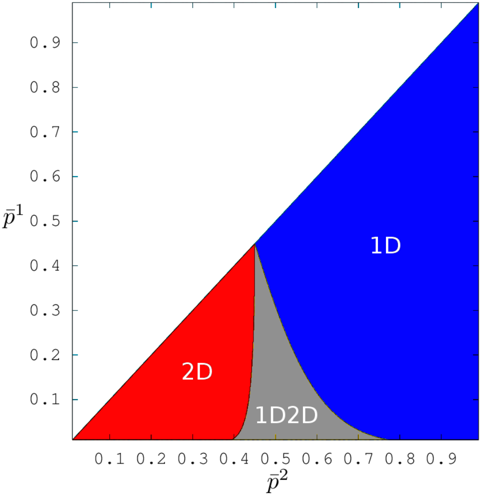
1D: agent 1 dominates; 2D: agent 2 dominates; 1,2S: both agents survive;

$\pi^* = 0.45, \gamma^1 = \gamma^2 = 0.5$



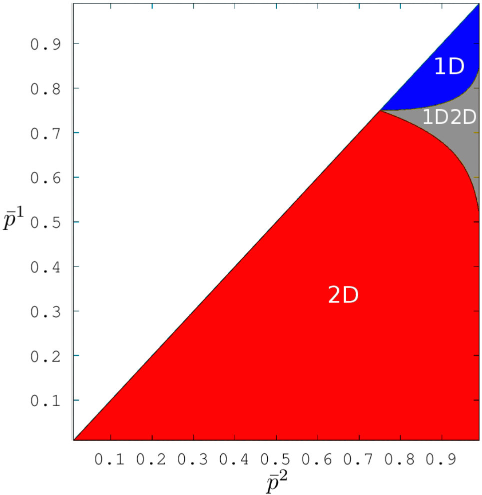
1D: agent 1 dominates; 2D: agent 2 dominates; 1D2D: either agent 1 or agent 2 dominates.

$\pi^* = 0.45, \gamma^1 = 2, \gamma^2 = 0.5$



1D: agent 1 dominates; 2D: agent 2 dominates; 1D2D: either agent 1 or agent 2 dominates.

$\pi^* = 0.75, \gamma^1 = 0.5, \gamma^2 = 2$



1D: agent 1 dominates; 2D: agent 2 dominates; 1D2D: either agent 1 or agent 2 dominates.

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Conclusion

We obtained generic sufficient conditions for the survival and dominance in a two-bettor parimutuel system applying Bottazzi and Dindo (2015).

The solution is easy and does not depend on fine details about bettors strategies.

We show that luck plays a generic role in deciding the ultimate winner (system is not always ergodic).

`http://www.lem.sssup.it/WPLem/2018-08.html`

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